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Fermionic order and disorder variables and correlation functions*

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Abstract. Order and disorder operators and correlation functions are studied in the context of a fermionic quantum field theory in $1+1$ dimensions.

The concept of order–disorder duality has its roots in the work of Krammers and Wannier [1] who advanced the idea that a complementarity should exist between the ordered and disordered phases of a given system. More recently, Kadanoff and Ceva [2] introduced the concept of a disorder variable and showed that these operators should obey certain commutation relations with the basic hamiltonian variables which are known as dual algebra. Later on, Fradkin *et al* [3] showed that the disorder operator should be the creation operator of the topological excitations eventually present in the system, arriving thereby at an interpretation of phase transitions as a condensation of topological excitations.

The idea of order–disorder duality was introduced in the framework of $(1+1)$-dimensional continuum quantum field theory in [4] where it was shown, in particular, that the Mandelstam soliton creation operator [5], the cornerstone of the bosonization method [5, 6, 9], could be treated as a product of order and disorder variables whose correlation functions could be obtained by generalizing the methods of Kadanoff and Ceva [2] for continuum field theory. In this way, the ideas of order–disorder duality [1, 2], soliton creation operators [3] and bosonization [5, 6] were unified in [4].

The order–disorder duality concept was also used in the full operator quantization of solitons [7] in two, three and four spacetime dimensions [8].

In the present work, we take the theory of a free massless fermionic field in $1+1$ dimensions and introduce the correlation functions of the order and disorder variables which may be identified in such a theory, according to the formulation established in [7]. These correlation functions turn out to be fermionic determinants, evaluated at certain given external fields A_μ. These determinants may be computed exactly and we show that the correlation functions of the composite order $\times$ disorder operator are identical to the massless Thirring field correlation functions [9]. We thereby re-obtain the equivalence of the massless Thirring model (MTM) to a free fermion, from a different point of view, which allows us to obtain of the exact solution for the correlation functions without having to resort to bosonization.

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From the form of the correlation functions, we infer expressions for the dual operators and show that they obey a dual algebra which generalizes that of [2] for continuum quantum field theory.

Given the Lagrangian

$$\mathscr{L} = \mathrm{i}\bar{\psi}\not{\partial}\psi \tag{1}$$

it follows that an order–disorder duality structure may be uncovered and which is embodied in the local order and disorder operators, respectively, $\sigma(x)$ and $\mu(x)$ [7]. The correlation functions of such dual objects were studied in [7] where it was shown that, in the case of the Lagrangian above, the Euclidean disorder correlation function is

$$\langle\mu(x)\mu^*(y)\rangle = Z^{-1}\int D\psi\, D\bar{\psi}\exp\left[-\int \mathrm{d}^2 z\, \mathrm{i}\bar{\psi}\not{D}\psi\right] = \det\left[\frac{\mathrm{i}\not{D}}{\mathrm{i}\not{\partial}}\right] \tag{2}$$

where $D_\mu = \partial_\mu - \mathrm{i}eA_\mu(z; x, y)$ and

$$A_\mu(z; x, y) = \int_{x,C}^{y} \varepsilon^{\mu\nu}\delta^2(z-\xi)\,\mathrm{d}\xi_\nu. \tag{3}$$

In the expression above, C is an arbitrary curve in Euclidean 2-space, connecting x and y. The correlation function $\langle\mu\mu^*\rangle$ is independent of C. This may be seen by performing the gauge transformation

$$A_\mu^C \to A_\mu^C + \partial_\mu\theta(R) = A_\mu^C - \oint_{C-C'} \varepsilon^{\mu\nu}\delta^2(z-\xi)\,\mathrm{d}\xi_\nu = A_\mu^{C'}. \tag{4}$$

In this expression, $\theta(R)$ is the two-dimensional Heaviside function with support in the region R, whose boundary is $\Gamma = C - C'$.

In analogous way, it was shown in [7] that the Euclidean order correlation function is given by

$$\langle\sigma(x)\sigma^*(y)\rangle = Z^{-1}\int D\psi\, D\bar{\psi}\exp\left[-\int \mathrm{d}^2 z\, \mathrm{i}\bar{\psi}\tilde{\not{D}}\psi\right] = \det\left[\frac{\mathrm{i}\tilde{\not{D}}}{\mathrm{i}\not{\partial}}\right] \tag{5}$$

where $\tilde{D}_\mu = \partial_\mu - \mathrm{i}\tilde{e}\tilde{A}_\mu(z; x, y)$ and

$$\tilde{A}_\mu(z; x, y) = -\mathrm{i}\varepsilon_{\mu\nu}A^\nu(z; x, y) = -\mathrm{i}\int_{x,C}^{y} \delta^2(z-\xi)\,\mathrm{d}\xi_\mu. \tag{6}$$

Since $\tilde{A}_\mu$ does not contain the $\varepsilon^{\mu\nu}$, it follows that path independence can no longer be obtained by the gauge transformation (4). In order to obtain a local σ field and, as a consequence, a path independent $\langle\sigma\sigma^*\rangle$ function, we will have to introduce, later on, a path-dependent renormalization counterterm.

In expressions (2) and (5), the constants e and $\tilde{e}$ appearing in D_μ and $\tilde{D}_\mu$ are real parameters on which the disorder and order operators depend.

The generalization of (2) and (5) for arbitrary correlation functions may be obtained straightforwardly, by just inserting additional external fields A_μ or $\tilde{A}_\mu$. One may obtain, for instance, the mixed Euclidean four-point function $\langle\sigma\sigma^*\mu\mu^*\rangle$ as

$$\langle\sigma(x)\sigma^*(y)\mu(x')\mu^*(y')\rangle = \det\left\{\frac{\mathrm{i}(\not{\partial} - \mathrm{i}[e\not{A}(z; x', y') + \tilde{e}\tilde{\not{A}}(z; x, y)])}{\mathrm{i}\not{\partial}}\right\}. \tag{7}$$

From (2) and (5), one can read the form of the dual operators σ and μ. In Minkowski space, we have

$$\begin{aligned}\sigma(x)&=\exp\left\{\mathrm{i}\tilde{e}\int \mathrm{d}^2z\,\tilde{j}^{\mu}(z)A_{\mu}(z;x)\right\}\\ \mu(x)&=\exp\left\{\mathrm{i}e\int \mathrm{d}^2z\,j^{\mu}(z)A_{\mu}(z;x)\right\}\end{aligned}\tag{8}$$

where

$$A_{\mu}(z;x)=\int_{x,c}^{\infty}\varepsilon_{\mu\nu}\delta^2(z-\xi)\,\mathrm{d}\xi\,\nu\tag{9}$$

and $j^{\mu}=\bar{\psi}\gamma^{\mu}\psi$ is the U(1) current and $\tilde{j}^{\mu}\equiv\varepsilon^{\mu\nu}j_{\nu}=\bar{\psi}\gamma^{\mu}\gamma^5\psi$ is the axial U(1) current. Choosing the curve c in (9) as a straight line going from x to ∞ along the ξ^1 axis and using the current algebra relation

$$[j^0(x,t),j^1(y,t)]=\mathrm{i}\partial_x\delta(x-y)\tag{10}$$

it is easy to show that

$$\mu(x,t)\sigma(y,t)=\exp\left(\mathrm{i}\frac{e\tilde{e}}{\pi}\theta(x-y)\right)\sigma(y,t)\mu(x,t).\tag{11}$$

This is the dual algebra which generalizes that of Kadanoff and Ceva [2] for continuum field theory [4, 7, 8].

The expression for the fermionic determinant appearing in (2), (5) and (7) is well known [10] and the result for the mixed function will be

$$\langle\sigma(x)\sigma^*(y)\mu(x')\mu^*(y')\rangle$$
$$=\exp\left\{-\frac{1}{2\pi}\int \mathrm{d}^2z(eA_{\mu}+\tilde{e}\tilde{A}_{\mu})\left[\delta^{\mu\nu}-\frac{\partial^{\mu}\partial^{\nu}}{\partial^2}\right](eA_{\nu}+\tilde{e}\tilde{A}_{\nu})\right\}.\tag{12}$$

The expression in the exponent in (12) contains four terms, namely $T_{e^2}+T_{\tilde{e}^2}+T_{e\tilde{e}}+T_{\tilde{e}e}$. Let us evaluate them. Using the fact that $\partial^{-2}=(1/2\pi)\ln|z|$ and equation (3), we may write the $\partial^{\mu}\partial^{\nu}$ part of T_{e^2} as

$$\frac{e^2}{(2\pi)^2}\int \mathrm{d}^2z\,\mathrm{d}^2z'\int_{x',c}^{y'}\mathrm{d}\xi_{\alpha}\int_{x',c}^{y'}\mathrm{d}\eta_{\beta}\,\varepsilon^{\mu\alpha}\varepsilon^{\nu\beta}\partial_z^{\mu}\partial_z^{\nu}\ln|z-z'|\delta^2(z-\xi)\delta^2(z'-\eta).\tag{13}$$

Expanding the product $\varepsilon^{\mu\alpha}\varepsilon^{\nu\beta}=\delta^{\mu\nu}\delta^{\alpha\beta}-\delta^{\mu\beta}\delta^{\nu\alpha}$ and using the fact that $\partial^2((1/2\pi)\ln|z-z'|)=\delta^2(z-z')$, we see that the first term in the expansion of the product of ε's cancels the $\delta^{\mu\nu}$ part of T_{e^2}. After integration over z and z' we are left, therefore, with

$$T_{e^2}=\frac{e^2}{(2\pi)^2}\int_{x',c}^{y'}\mathrm{d}\xi_{\nu}\int_{x',c}^{y'}\mathrm{d}\eta_{\mu}\,\partial_{\xi}^{\nu}\partial_{\eta}^{\mu}\ln|\xi-\eta|$$

or

$$T_{e^2}=-\frac{e^2}{2\pi^2}[\ln|x'-y'|-\ln|\varepsilon|].\tag{14}$$

In arriving at (14), we used the fact that $\partial_{\xi}=-\partial_{\eta}$ acting on $\ln|\xi-\eta|$. In the last expression, ε is a short distance cut-off. At the end we will take the limit $\varepsilon\to 0$.

Let us consider $T_{\tilde{e}^2}$, now. Using (6) and following the same reasoning which led us to (14), we immediately get

$$T_{\tilde{e}^2}=-\frac{\tilde{e}^2}{2\pi^2}[\ln|x-y|-\ln|\varepsilon|]+\frac{\tilde{e}^2}{2\pi}\int_{x,c}^{y}\int_{x,c}^{y}\delta^2(\xi-\eta)\,\mathrm{d}\xi_\mu\,\mathrm{d}\eta^\mu. \tag{15}$$

Let us evaluate now the crossed terms $T_{e\tilde{e}}$. Since $\partial_\mu\partial_\nu\ln|z-z'|=\partial'_\mu\partial'_\nu\ln|z-z'|$, it follows that $T_{e\tilde{e}}=T_{\tilde{e}e}$. We therefore have

$$\begin{aligned}T_{e\tilde{e}}+T_{\tilde{e}e}=\mathrm{i}\frac{e\tilde{e}}{\pi}\mathrm{d}^2z\int_{x',c}^{y'}\mathrm{d}\xi_\alpha\int_{x,c}^{y}\mathrm{d}\eta_\mu\,\varepsilon^{\mu\alpha}\delta^2(z-\xi)\delta^2(z-\eta)\\ -\mathrm{i}\frac{e\tilde{e}}{2\pi^2}\int\mathrm{d}^2z\,\mathrm{d}^2z'\int_{x',c}^{y'}\mathrm{d}\xi_\alpha\int_{x,c}^{y}\mathrm{d}\eta_\nu\,\varepsilon^{\mu\alpha}\partial_\mu^z\partial_\nu^z\ln|z-z'|\delta^2(z-\xi)\delta^2(z-\eta).\end{aligned} \tag{16}$$

The first term in the above expression vanishes,

$$\mathrm{i}\frac{e\tilde{e}}{2\pi}\int_{x',c}^{y'}\mathrm{d}\xi_\alpha\int_{x,c}^{y}\mathrm{d}\eta_\mu\,\varepsilon^{\mu\alpha}\delta^2(\xi-\eta)=0 \tag{17}$$

because the only contributions come from the points where $\xi=\eta$, due to the delta but at these points $\mathrm{d}\boldsymbol{\xi}\equiv\mathrm{d}\boldsymbol{\eta}$ and the vector product is zero.

Integrating over z and z' in the second term in (16) and using the Cauchy-Riemann equation

$$\varepsilon^{\mu\alpha}\partial_\mu^\xi\ln|\xi-\eta|=\partial_\xi^\alpha\arg(\xi-\eta) \tag{18}$$

we get

$$T_{e\tilde{e}}+T_{\tilde{e}e}=\mathrm{i}\frac{e\tilde{e}}{2\pi^2}\int_{x',c}^{y'}\mathrm{d}\xi_\alpha\int_{x,c}^{y}\mathrm{d}\eta_\nu\,\partial_\xi^\alpha\partial_\eta^\nu\arg(\xi-\eta) \tag{19}$$

where again we used the fact that $\partial_\eta=-\partial_\xi$ acting on $\arg(\xi-\eta)$. The remaining integrals may now be performed straightforwardly yielding the result

$$T_{e\tilde{e}}+T_{\tilde{e}e}=\mathrm{i}\frac{e\tilde{e}}{2\pi^2}[\arg(x'-x)+\arg(y'-y)-\arg(x'-y)-\arg(y'-x)]. \tag{20}$$

Observe that the disorder correlation function is $\langle\mu\mu^*\rangle=\exp[T_{e^2}]$ and the other correlation function is $\langle\sigma\sigma^*\rangle=\exp[T_{\tilde{e}^2}]$. T_{e^2} possesses a short distance divergence which may be eliminated by introducing the renormalized operator

$$\mu_R(x)=\mu(x)\exp\left(-\frac{e^2}{4\pi^2}\ln|\varepsilon|\right). \tag{21}$$

This immediately leads us to the renormalized disorder correlation function

$$\langle\mu(x)\mu^*(y)\rangle_R=\frac{1}{|x-y|^{e^2/2\pi^2}}. \tag{22}$$

$T_{\tilde{e}^2}$ possesses, in addition to the short distance divergence $\ln|\varepsilon|$, a path-dependent divergent term, which was to be expected. As we observed, $\langle\sigma\sigma^*\rangle$ was not path independent. Using the properties of the delta function and writing $\int_x^y=\int_x^\infty-\int_y^\infty$, one may see that

$$\int_{x,c}^{y}\mathrm{d}\xi_\mu\int_{x,c}^{y}\mathrm{d}\eta^\mu=\int_{x,c}^{\infty}\mathrm{d}\xi_\mu\int_{x,c}^{\infty}\mathrm{d}\eta^\mu-\int_{y,c}^{\infty}\mathrm{d}\xi_\mu\int_{y,c}^{\infty}\mathrm{d}\eta^\mu. \tag{23}$$

Defining the renormalized σ field as

$$\sigma_R(x)=\sigma(x)\exp\left\{-\frac{\tilde{e}^2}{4\pi^2}\ln|\varepsilon|-\frac{\tilde{e}^2}{4\pi}\int_{x,c}^{\infty}\mathrm{d}\xi_\mu\int_{x,c}^{\infty}\mathrm{d}\eta^\mu\,\delta^2(\xi-\eta)\right\}\tag{24}$$

we eliminate all divergences from $\langle\sigma\sigma^*\rangle$, obtaining the path-independent renormalized function

$$\langle\sigma(x)\sigma^*(y)\rangle_R=\frac{1}{|x-y|^{\tilde{e}^2/2\pi^2}}.\tag{25}$$

From the large distance behaviour in (22) and (25), we see that $\langle\mu\rangle_R=\langle\sigma\rangle_R=0$ as it should be in a theory without mass gap [11].

Collecting the terms T_{e^2}, $T_{\tilde{e}^2}$, $T_{e\tilde{e}}+T_{\tilde{e}e}$ and taking into account the above renormalizations of σ and μ, we arrive at the following expression for the mixed four-point function:

$$\begin{aligned}&\langle\sigma(x)\sigma^*(y)\mu(x')\mu^*(y')\rangle_R\\&\quad=\frac{1}{|x'-y'|^{e^2/2\pi^2}}\frac{1}{|x-y|^{\tilde{e}^2/2\pi^2}}\\&\qquad\times\exp\mathrm{i}\frac{e\tilde{e}}{2\pi^2}\Big[\arg(x'-x)+\arg(y'-y)-\arg(x'-y)-\arg(y'-x)\Big].\end{aligned}\tag{26}$$

Since the arg functions are defined up to 2π factors, we see that the above correlation function contains the ambiguous multiplicative factors $\mathrm{e}^{\mathrm{i}(e\tilde{e}/\pi)n}$. These are just a reflex of the dual algebra (11) and are associated with the various orderings of operators in the LHS of (26) [4].

As a consequence of (11), we see that the composite fields $\psi_1(x)=\lim_{x\to x'}\sigma(x)\mu(x')$ and $\psi_2=\lim_{x\to x'}\sigma^*(x')$ obey the generalized statistics

$$\psi_i(x,t)\psi_i(y,t)=\exp\left(\mathrm{i}\frac{e\tilde{e}}{\pi}\varepsilon(x-y)\right)\psi_i(y)\psi_i(x)\tag{27}$$

with spin $S=e\tilde{e}/2\pi^2$. Taking the limits $x\to x'$ and $y\to y'$ or $y\to x'$ and $x\to y'$ in (26) and regularizing the arg functions as $\lim_{\varepsilon\to 0}\arg(\varepsilon)=0$, we immediately get the ψ correlation functions

$$\langle\psi_{1(2)}(x)\psi^+_{1(2)}(y)\rangle=|x-y|^{-(e^2+\tilde{e}^2)/2\pi^2}\exp\{-(+)\mathrm{i}s[\arg(x-y)+\arg(y-x)]\}.\tag{28}$$

These are precisely the Euclidean version of the Klaiber solution [9] for the MTM correlation functions and therefore we identify ψ_i with the Thirring field. The dimension of ψ is $d=(e^2+\tilde{e}^2)/4\pi^2$. Higher functions may be obtained straightforwardly by inserting additional A_μ and $\tilde{A}_\mu$ fields. The coupling constant of the MTM is related to e and $\tilde{e}$ as $g=\pi(1-e/\tilde{e})$.

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